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NOTE ON THE MEAN-AREA OF THE PRISMOID AND SOME ASSOCIATED THEOREMS.

By Prof. W. H. Echols, Charlottesville, Va.

1. A prismoid is here defined to be the solid which is cut out between two fixed parallel planes by a right line moving according to any law, which however finally returns to its initial position.

The closed curves traced in the fixed planes are called the bases of the prismoid. The section of the solid by any plane parallel to the base planes is termed a " cross-section " or simply a " section."

If a second straight line passes through a fixed point in one of the base planes and moves so as to be always parallel to the first straight line, it cuts out a cone between the base planes. This cone is called the " director-cone " of the prismoid. Evidently the geometrical figure of the prismoid is completely determined when the bases and director-cone are known.

2. The " mean-area " of the prismoid, is that *area* whose product by the length of the prismoid (the distance between the planes of the bases) is the volume of the solid. In other words, it is the average or mean value of all the cross-sections of the solid.

The most familiar expression giving the mean-area, is that due to Newton, the well known so-called " prismoid-formula,"

$$\Omega = \tfrac{1}{6}(B_1 + B_2 + 4M) \qquad (1)$$

wherein B_1 and B_2 are the areas of the bases and M the area of the section mid-way between the bases. This is, however, not a prismoid-formula ; since it is the expression for the mean-area of all solids whose cross-section areas are cubic functions of their lengths, while the prismoid is a solid whose sectional area is a quadratic function of its length, consequently since (1) is applicable to solids of higher order than prismoids, we reject this expression as a " prismoid-formula."

The expression for the mean-area

$$\Omega = \tfrac{1}{2}(B_1 + B_2) - \tfrac{1}{6}A\,, \qquad (2)$$

wherein A is the area of the base of the associate cone, is, I believe, due to Koppe and seems to be the first prismoid-formula found. This formula in connection with (1) easily furnishes the value

$$\Omega = M + \tfrac{1}{12}A\,. \qquad (3)$$

In this form the mean area has been used by engineers for computing earth-work volumes. The writer's first acquaintance with it thus employed was in the lectures to engineering students at the University of Virginia in 1880. This is a two-term formula, but it involves an extraneous area, that of the director-cone base, which is not a section of the solid.

In 1881, Dr. Halsted published the formula

$$\begin{aligned} Q &= \tfrac{1}{4}\,(B_1 + 3\,S_{\frac{2}{3}}) \\ &= \tfrac{1}{4}\,(3\,S_{\frac{1}{3}} + B_2)\,, \end{aligned} \tag{4}$$

wherein $S_{\frac{2}{3}}$, $S_{\frac{1}{3}}$ are the areas of sections distant $\frac{2}{3}$ and $\frac{1}{3}$ the length of the solid respectively from the end B_1 towards B_2. In his *Mensuration*, p. 130, Ed. 1890, the author remarks upon this formula as being "the first and only two-term prismoidal formula ever printed."

It is the object of the present note to show that there are an unlimited number of such two-term formulæ, involving sections of the prismoid only.

3. Consider a prismoid, project it orthogonally on a plane parallel to the bases. Let the straight line B_1B_2 be the projection of the straight line which generates the curved surface of the prismoid, the points B_1 and B_2 being the projections of the points where the straight line intersects the planes of the bases. Evidently any plane parallel to the bases cuts this generator in a point so that the intercepts between this plane and the bases are in constant ratio, as are also their corresponding projections. Let P be the projection on the line B_1B_2 of the point in which the generator pierces any such third plane. The point P divides the line B_1B_2 (externally or internally as the case may be) in constant ratio, say m/n, so that if $B_1B_2 = l$, then $B_1P = ml/(m+n) = \mu l$ and $PB_2 = nl/(m+n) = \nu l$.

As the generating line cuts the prismoid out between the planes of its bases, the points B_1 and B_2 trace closed curves equal and parallel to these bases in the plane of projection, and the cross-section traced in the plane parallel to the bases is equal to the curve traced by P. Consequently we need only consider the figures in the plane of projection. To express the area of any cross-section whatever, in terms of the bases is then merely to quote Elliott's extension of Holditch's theorem, which may be briefly deduced as follows:—

Let the line B_1B_2 move through the angle $d\theta$ to a consecutive position. Let λ be the distance from B_1 to the contact of the moving line with its envelope. Then, indicating areas by capitals, we have

$$\begin{aligned} dB_2 - dB_1 &= \tfrac{1}{2}\,(\lambda + l)^2\,d\theta - \tfrac{1}{2}\,\lambda^2\,d\theta \\ &= \lambda l\,d\theta + dA\,, \end{aligned}$$

writing dA for $\frac{1}{2} l^2 d\theta$. Also

$$dP - dB_1 = \tfrac{1}{2}(\lambda + \mu l)^2 d\theta - \tfrac{1}{2}\lambda^2 d\theta$$
$$= \mu\lambda l\, d\theta + \mu^2 dA .$$

Whence, at once by elimination of $\lambda l\, d\theta$, remembering that $\mu + \nu = 1$,

$$dP = \nu dB_1 + \mu dB_2 - \mu\nu dA .$$

Integrating for a complete circuit, that is from the initial to the final position of the line, we have

$$P = \nu B_1 + \mu B_2 - \mu\nu A ,$$

wherein P is the area of the cross-section dividing the interval between the planes of the bases (externally or internally) in the ratio m/n, B_1 and B_2 are the areas of the bases and $A = \int \frac{1}{2} l^2 d\theta$ is the area of the base of the director-cone.

4. It is not necessary in the general treatment that we should consider B_1 and B_2 to be the extreme sections (or bases) of the prismoid; so let us consider B_1 and B_2, as above, to be merely S_1 and S_2, any selected reference sections distant apart h, and proceed to determine the mean area of a prismoid whose bases are at distances from S_1 equal to xh and yh respectively, whose length will then be $(y - x)H$, x and y being respectively the inferior and superior limits of μ.

We have for the mean-area between these limits

$$\Omega = \frac{1}{y-x}\int_x^y P\, d\mu$$
$$= \frac{1}{y-x}\int_x^y (\nu S_1 + \mu S_2 - \nu\mu A_h)\, d\mu$$
$$= \frac{1}{y-x}\int_x^y [S_1 + (S_2 - S_1 - A)\mu + \mu^2 A_h]\, d\mu$$
$$= \left[1 - \frac{x+y}{2}\right] S_1 + \frac{x+y}{2} S_2 + \left[\frac{x^2+xy+y^2}{3} - \frac{x+y}{2}\right] A_h , \qquad (5)$$

which is a symmetrical function of the limits.

Those real values of x and y which satisfy

$$x^2 + xy + y^2 - \tfrac{3}{2}(x + y) = 0 , \qquad (6)$$

give *two-term* prismoidal formulæ in terms of the cross-sections S_1 and S_2. The cone base A_h corresponds to the sections S_1 and S_2 having altitude equal to the distance between their planes. If this be h and H, the altitude of the prismoid whose mean area we have found, then the base of the corresponding director-cone is determined by

$$\frac{A_h}{A_H} = \left[\frac{h}{H}\right]^2 = \frac{1}{(y-x)^2}.$$

The above equality is the equation to an ellipse referred to rectangular axes of x and y. The center is $x = \frac{1}{2}$, $y = \frac{1}{2}$, and the equation when transferred to center and axes turned through $\pi/4$ is

$$\frac{x^2}{1/2} + \frac{y^2}{3/2} = 1.$$

The maximum and minimum values of y in (6) are at the points ($x = 0$, $y = 3/2$), ($x = 1$, $y = -1/2$), and those of x are the same interchanging x and y. Therefore, for any value of x whatever between $-1/2$ and $+3/2$ inclusive, we have a section S_1 with a corresponding section S_2, furnished by the y corresponding to x in (6), such that the mean-area of the prismoid is given by the two-termed formula*

$$\Omega = \left[1 - \frac{x+y}{2}\right] S_1 + \frac{x+y}{2} S_2. \tag{7}$$

Dr. Halsted's formula is the particular case furnished by putting in the extreme values of x or y. Thus if $x = -\frac{1}{2}$ and $y = +1$, we have

$$\Omega = \tfrac{3}{4} S_1 + \tfrac{1}{4} S_2, \tag{8}$$

and now S_2 is the base B_2 and S_1 the cross-section at $\frac{2}{3}$ the length from S_2. The only other case which seems worthy of notice, is that in which S_1 and S_2 are the cross-sections which are equidistant from the ends of the prismoid. Thus if H be the length of a prismoid it is easy to see that if we impose the condition that $y - 1 = x$, we derive the mean-area formula

$$\Omega = \tfrac{1}{2}(S_1 + S_2), \tag{9}$$

wherein S_1 and S_2 are distant from the ends an amount

$$\frac{\sqrt{3}-1}{2\sqrt{3}} H = 0.2114\, H.$$

* Those values of x between $-\frac{1}{2}$ and 0 (inclusive) furnish all of the two-term formulæ, wherein S_1 and S_2 are actual sections of the prismoid between its bases.

This may be regarded then as the simplest of all the *two-term* formulæ.

A general formula in terms of sections equidistant from the ends may at once be written by putting $x = -q$, $y = 1 + q$, whence

$$\mathcal{Q} = \tfrac{1}{2}(S_1 + S_2) + \left[\frac{1 + q + q^2}{3} - \tfrac{1}{2}\right]\frac{A_H}{(1+2q)^2}. \tag{10}$$

Particular cases are

$$(q = 1) \qquad \mathcal{Q} = \tfrac{1}{2}(S_1 + S_2) + \tfrac{1}{8} A_H, \tag{11}$$

in which the sections are distant $\frac{1}{3} H$ from the ends.

$$(q = \tfrac{1}{2}) \qquad \mathcal{Q} = \tfrac{1}{2}(S_1 + S_2) + \tfrac{1}{48} A_H, \tag{12}$$

the sections being distant $\frac{1}{4} H$ from the ends.

$$(q = \tfrac{1}{3}) \qquad \mathcal{Q} = \tfrac{1}{2}(S_1 + S_2) - \tfrac{1}{150} A_H, \tag{13}$$

the sections being distant $\frac{1}{5} H$ from the ends.

Evidently as q becomes smaller the formula converges to Koppe's old form (2).

In like manner we may write a formula involving the mid-section and one base. For, let $x = -a$, $y = +a$; then

$$\mathcal{Q} = M + \frac{a^2}{3} A_h$$

$$= M + \tfrac{1}{12} A_H,$$

the old form (3).

5. Any two-term formula (7) being independent of the director-cone, gives the following curious theorem :—

A straight line moving so as to intersect the boundaries S_1 and S_2, from an initial position back to that position, cuts out between the planes of the bases of that prismoid a volume which is always the same whatever be the manner in which the line moves between its initial and final position. Thus all prismoids (7) whose curved surfaces pass through the boundaries of S_1 and S_2 and whose bases are in the planes determined by x and y, have the same volume.

6. As in § 3, consider the line B_1PB_2. From any point in its plane O draw $OB_1 = \gamma_1$, $OB_2 = \gamma_2$, $OP = \rho$. Then, letting $B_1B_2 = l$, we have, by a theorem of elementary geometry

$$\rho^2 = \nu\gamma_1^2 + \mu\gamma_2^2 - \nu\mu l^2,$$

and the above theorems of mean-value hold true with regard to the mean value of ρ^2, mutatis mutandis.

7. Again, as the line B_1B_2 turns through $d\theta$, the points B_1, B_2, and P describe elementary arcs, ds_1, ds_2, and $d\sigma$, and it may easily be shown that

$$(d\sigma)^2 = \nu(ds_1)^2 + \mu(ds_2)^2 - \nu\mu(da)^2 ,$$

wherein da is the corresponding elementary arc of the boundary of the base of the director-cone. The theorems of mean-value apply also to the mean-value of $(d\sigma)^2$, with the necessary changes of interpretation.

8. Let us apply the same method to determine the mean ordinate to the parabola

$$y = x^2 + 2ax + b ,$$

in any interval L taken on the x-axis.

Let y_1 and y_2 be any ordinates distant apart l. Let y be any ordinate which divides the interval l in the ratio m/n so that the abscissa of y is $x_1 + \mu l$ or $x_2 - \nu l$. We have

$$\begin{aligned} y &= (x_1 + \mu l)^2 + 2a(x_1 + \mu l) + b \\ &= y_1 + 2\mu l x_1 + \mu^2 l^2 + 2a\mu l , \\ y &= (x_2 - \nu l)^2 + 2a(x_2 - \nu l) + b \\ &= y_2 - 2\nu l x_2 + \nu^2 l^2 - 2a\nu l . \end{aligned}$$

Eliminating a between these we get

$$\begin{aligned} y &= \nu y_1 + \mu y_2 - \mu\nu l^2 \cdot \\ &= y_1 + (y_2 - y_1 - l^2)\mu + \mu^2 l^2 \end{aligned}$$

Determining the mean value of y in the interval $x_1 + pl$, $x_1 + ql$, or $L = (p - q)l$, we have

$$\begin{aligned} y_m &= \frac{1}{p-q}\int_q^p [y_1 + (y_2 - y_1 - l^2)\mu + \mu^2 l^2]\, d\mu \\ &= \left[1 - \frac{p+q}{2}\right] y_1 + \frac{p+q}{2} y_2 + \left[\frac{p^2 + pq + q^2}{3} - \frac{p+q}{2}\right] \frac{L^2}{(p-q)^2} , \quad (14) \end{aligned}$$

in which the square on the interval L is the area associated with the parabola in the same manner that the base of the associate cone is associated with the prismoid. For example, we eliminate L by giving p and q, values which satisfy

$$p^2 + pq + q^2 - \tfrac{3}{2}(p + q) = 0 ,$$

obtaining the mean ordinate in interval L, in terms of two ordinates.

In (14) put $p = +c$, $q = -c$, we have

$$y_m = y_1 + \tfrac{1}{12} L^2 ,$$

wherein y_1 is the ordinate at the middle of the interval L. This remarkable little formula I have not seen elsewhere, surely it must be known. From it follows a mensuration rule. If the area of a curve be desired throughout an interval L and L be divided into n irregular intervals $l_1, \ldots, l_n$ and the ordinates at the middle of these intervals by $y_1, \ldots, y_n$, then the area is

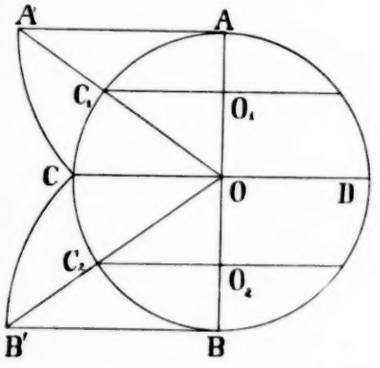
$$\Omega = \sum_1^n y_r l_r + \tfrac{1}{12} \sum_1^n l_r^3 ,$$

which is exact when the curves are parabolæ throughout the subintervals. If the subintervals are equal,

$$\Omega = l \Sigma y_r + \tfrac{1}{12} n l^3 .$$

As an example of theorems which flow from these results we may enunciate the following :—

Let AB, CD be the right-angled diameters of the great circle of a sphere with center O. Draw the tangents at A and B. With A and B as centers and radii AC and BC draw arcs cutting the tangents in A' and B'. Draw $A'O$ and $B'O$ cutting the circumference in C_1 and C_2. Let (C_1) and (C_2) be the circles on the sphere through C_1 and C_2 with centers on AB. Then,



Any right line moving on the circles (C_1) and (C_2), starting from any given position and returning to that position, cuts out a solid between the tangent planes at A and B whose volume equals that of the sphere, whatever may be the manner in which the line moves between its initial and terminal position. Particular cases are, the cone with vertex O, the cylinder, and the hyperboloids of revolution. This theorem is an interesting generalization of the famous old Archimedian theorem of the cylinder and the sphere.

9. One of the most familiar as well as the most useful of the theorems of mean value is Newton's theorem for the cubic (Methodus Differentialis) which is given in (1). Let us apply this method to the generalization of this theorem.

Let

$$y = x^3 + ax^2 + bx + c \qquad (15)$$

be any cubic function. Let (x_1y_1), (x_2y_2), (x_3y_3) be any fixed values. Let $x_2 - x_1 = l_1$, $x_3 - x_2 = l_2$. Let x and y be current values and let

$$\begin{aligned} x &= x_1 + pl_1 = x_2 + ql_1 \\ &= x_3 + sl_2 = x_2 + rl_2 . \end{aligned}$$

Form four equations by substituting these values for x in (15), and eliminate a and b from them. Whence results

$$\begin{aligned} (pql_1^2 - rsl_2^2)\, y = {} & rpql_1^2\, y_3 + rsql_2^2\, y_1 - (pqsl_1^2 + rspl_2^2)\, y_2 \\ & - rspql_1^2 l_2^2 (l_1 + l_2) . \end{aligned} \tag{16}$$

Remembering that $p - q = 1$, $r - s = 1$, and $ql_1 = rl_2$, (16) becomes

$$y = rpk_1 y_3 + rsk_2 y_1 - psy_2 - psql_1^2 l_2 , \tag{17}$$

wherein we have written $k_1 = l_1/(l_1 + l_2)$ and $k_2 = l_2/(l_1 + l_2)$ for brevity. Eliminating r, p, and s and writing $\rho = l_1/l_2$, we have

$$\begin{aligned} y = {} & (q + q^2)\, \rho k_1\, y_3 + (\rho q^2 - q)\, \rho k_2\, y_1 - (\rho q^2 + \overline{\rho - 1}\, q - 1)\, y_2 \\ & - (\rho q^3 + \overline{\rho - 1}\, q^2 - q)\, l_1^2 l_2 . \end{aligned}$$

The mean value of y between the limits $q = u$ and $q = v$, is

$$\begin{aligned} y_m = {} & \frac{1}{u - v} \int_v^u y\, dq \\ = {} & (\tfrac{1}{3} S_2 + \tfrac{1}{2} S_1)\, \rho k_1 y_3 + (\tfrac{1}{3} \rho S_2 - \tfrac{1}{2} S_1)\, \rho k_2 y_1 - (\tfrac{1}{3} \rho S_2 + \overline{\rho - 1}\, \tfrac{1}{2} S_1 - 1)\, y_2 \\ & - (\tfrac{1}{4} \rho S_3 + \overline{\rho - 1}\, \tfrac{1}{3} S_2 - \tfrac{1}{2} S_1)\, l_1^2 l_2 \end{aligned} \tag{18}$$

wherein

$$\begin{aligned} S_3 &= u^3 + u^2 v + uv^2 + v^3 , \\ S_2 &= u^2 + uv + v^2 , \\ S_1 &= u + v . \end{aligned}$$

We may write (18)

$$y_m = A\rho k_1 y_3 + B\rho k_2 y_1 + Cy_2 + Dl_1^2 l_2 .$$

If we give u and v the values which satisfy the cubic equation

$$D = 0 ,$$

in u and v. Then we have three-term formulæ for the mean ordinate of the cubic. If we give to u and v the values where the cubic crosses the second

degree curve $A = 0$, $B = 0$, or $C = 0$, then we derive two-term formulæ for the mean ordinate of the cubic.

In particular, put $\rho = 1$, $k_1 = k_2 = \frac{1}{2}$ and $u = a$, $v = -a$. Then

$$y_m = \tfrac{1}{6}a^2 (y_1 + y_3) + (1 - \tfrac{1}{3} a^2)\, y_2 .$$

If $a = 1$, we derive Newton's formula

$$y_m = \tfrac{1}{6} (y_1 + y_3 + 4y_2) ,$$

and if we put $a^2 = 3$ we derive the remarkable formula

$$y_m = \tfrac{1}{2} (y_1 + y_3) .$$

But in this L the length between the limits is $2al$ and the distance of x_1 and x_3 from the limits is

$$\frac{a-1}{2a} L = \frac{\sqrt{3}-1}{2\sqrt{3}} L .$$

Therefore the same formula which we found for the quadratic (9) holds as a two-term formula for the cubic. We may say then, in regard to the theorem at the end of § 8, that not only do all ruled surfaces passing through the circles (C_1) and (C_2) cut out the volume of the sphere between the tangent planes at A and B, but so also do all cuboids passing through these circles.

10. Writers of elementary text-books on mensuration are too apt to forget that Gauss and Jacobi have touched this subject and that there is little chance for anything new. The best that we can do is to follow behind them and pick up such trifles as they have overlooked. Gauss has demonstrated (Werke III, 203) that the area of any rational integral function of degree $2n - 1$ may be expressed in terms of n properly chosen ordinates. The position of these ordinates having been shown by Jacobi to have for abscissæ the roots of the equation

$$\left[\frac{d}{dx}\right]^n [(x-p)(x-q)]^n = 0 ,$$

the area interval being $x = p$, $x = q$. The substitution of these roots in Lagrange's interpolation formula and subsequent integration produces the desired result.* Applying this method to the few simple cases, we have

1°. Cubic. $2n - 1 = 3$, $n = 2$. If u and v be the limits then

$$\left[\frac{d}{dx}\right]^2 (x-u)^2 (x-v)^2 = 0$$

gives

$$x = \tfrac{1}{2} (u + v) \pm \frac{u-v}{2\sqrt{3}} ,$$

* See Boole's Finite Differences, p. 53.

and the mean ordinate between u and v is

$$y_m = \tfrac{1}{2}(y_1 + y_2),$$

verifying the result in § 9.

2°. Quintic. $2n - 1 = 5$, $n = 3$,

$$\left[\frac{d}{dx}\right]^3 (x-u)^3 (x-v)^3,$$

gives as the position of the ordinates,

$$x = \tfrac{1}{2}(u+v),$$

$$x = \tfrac{1}{2}(u+v) \pm \tfrac{1}{2}(u-v)\sqrt{\tfrac{3}{5}},$$

corresponding to y_2, y_3, y_1, respectively, and the mean ordinate between u and v is

$$y_m = \tfrac{5}{18}(y_1 + y_3 + \tfrac{8}{5} y_2).$$

3°. The mean ordinate of the seventhic is interesting, we have

$$\left[\frac{d}{dx}\right]^4 (x-u)^4 (x-v)^4 = 0$$

furnishing

$$x = \tfrac{1}{2}(u+v) \pm \tfrac{1}{2}(u-v)\sqrt{\tfrac{3}{7} \pm \sqrt{\tfrac{168}{1715}}},$$

and the mean ordinate is

$$y_m = \tfrac{1}{4}(y_1 + y_2 + y_3 + y_4) - \tfrac{1}{42}\sqrt{\tfrac{1715}{168}}(y_1 - y_2 + y_4 - y_3),$$

y_1 and y_4 being the extreme ordinates. The striking resemblance of this to the ordinary rule with Poncelets' correction is noticeable, for $\sqrt{1715}/42\sqrt{168} = 1/13.145$ differs but little from Poncelets' 1/12, while the spacing of the ordinates 0.07, 0.33, 0.67, 0.93 does not differ much from Poncelets' 1/8, 3/8, 5/8, 7/8.

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TRANSFORMATION GROUPS IN SPACE OF FOUR DIMENSIONS.

By Dr. J. M. Page, Cobham, Va.

In a previous article* a number of the most important definitions used in Lie's Theory of Groups were introduced. In this article those definitions are used without further explanation.

In another paper† the writer ventured, without proof, the assertion that he had found *all* the primitive groups in space of four dimensions. It is proposed to prove a part of that assertion by showing that *none of the groups which leave invariant two different manifoldnesses, each of two dimensions, in space of four dimensions, can be primitive.*

At the same time a most important part of the Theory of Groups will be illustrated by showing how groups which fulfil certain given conditions may be determined in the most general manner possible.

It will be necessary, in the first place, to introduce a few additional definitions.

By reference to the former paper, the reader will readily see that the symbol of an infinitesimal transformation in the four variables x_1, x_2, x_3, x_4 must have the form

$$Xf = \xi_1(x_1, x_2, x_3, x_4)\frac{\partial f}{\partial x_1} + \xi_2()\frac{\partial f}{\partial x_2} + \xi_3()\frac{\partial f}{\partial x_3} + \xi_4()\frac{\partial f}{\partial x_4} = \sum_1^4 i\, \xi_i()\frac{\partial f}{\partial x_i}.$$

Here the ξ_i's are, as usual, *analytical* functions of their arguments in Weierstrass's sense of the word; and hence, if $x_k^{(0)}$ is a point of ordinary position the ξ_i can be expanded in convergent powers of $(x_k - x_k^{(0)})$, at least for regions lying very near $x_k^{(0)}$. If this analytical expansion happens to begin with a term of the rth power, the infinitesimal transformation is said to be of the rth order. The first term of an expansion of this kind is called the *initial* term.

Two groups in n variables, each of r members, are said to be *similar*, when by a proper choice of the independent variables the one group can be transformed into the other. All groups, therefore, which are similar to a known group, may be considered known. Also, it has been proved by Lie that if a group can be assigned which has the same setting as a required

* Annals of Mathematics, Vol. VIII, No. 4, "On Transformation Groups."

† American Journal of Mathematics, Vol. X, No. 4, "On the Primitive Groups of Transformations in Space of Four Dimensions."

group, and if the initial terms of the respective transformations of the two groups are the same, then the two groups are *similar*. In practical calculations, therefore, only the initial terms of the transformations will be needed.

The introduction of certain properly chosen linear combinations of the transformations of a group, in place of the given transformations, often simplifies the "bracket operations" between the transformations of the group, by removing the indeterminate constants which may occur in the results of those operations. The performance of this simplification is called *normalizing* the transformations of the group.

Analogous to the definition of an *imprimitive* group in the plane, it is easy to see that a group in space of n dimensions is *imprimitive*, when it leaves invariant a family of ∞^{n-q} manifoldnesses, each of q dimensions, which fill the space of n dimensions exactly once.

The customary abbreviation p_i for the partial differential coefficients $\frac{\partial f}{\partial x_i}$ will be used.

I.—Introductory.

1. *Transformation of zero and first order.*

If

$$x_1 = 0\,, \qquad x_2 = 0$$

and

$$x_3 = 0\,, \qquad x_4 = 0$$

are the two manifoldnesses which are invariant under the transformations of our group, it has been shown* that the transformations of the zero and of the first orders, respectively, of all possibly primitive groups, must have the following forms (writing, as usual, only the initial terms):

$$p_1,\ p_2,\ p_3,\ p_4\,;$$

$$x_1p_2\,,\ x_1p_1 - x_2p_2\,,\ x_2p_1\,;\quad x_3p_4\,,\ x_3p_3 - x_4p_4\,,\ x_4p_3\,; \qquad (S_4)$$

$$x_1p_1 + x_2p_2 - x_3p_3 - x_4p_4\,; \qquad (T)$$

with which transformations there *may* also occur

$$x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4 = \sum_1^4{}_i\, x_ip_i\,. \qquad (U)$$

2. *No transformation of an order higher than the second can occur.*

For, let s be the maximum order, so that a transformation of the order

$$Xf = \sum_1^4{}_i\, \xi_i^{(s)}(x_1, x_2, x_3, x_4)\, p_i$$

* American Journal of Mathematics, Vol. X, No. 4, p. 310.

occurs, where the index s shows that the $\xi_i^{(s)}$ begin with terms of the sth degree.

Since in all of the groups x_1, x_2 will be equally privileged with x_3, x_4, it may be assumed that $\xi_1^{(s)}$ and $\xi_2^{(s)}$ are not both zero. If now $\xi_1^{(s)}$ is not identically zero, it is seen, by properly combining Xf with p_1, p_2, p_3, and p_4 that

$$\xi_1^{(s)} \equiv \xi_1^{(s)}(x_1, x_2),$$

for the results of the bracket operations must always be capable of being linearly expressed in terms of the p_i, S_i, T, and U.

Furthermore, by combining Xf properly with $x_1 p_2$, it is seen that a transformation of the form

$$Yf \equiv x_1^{(s)} p_1 + y_2^{(s)} p_2 + y_3^{(s)} p_3 + y_4^{(s)} p_4$$

must occur in the group. Combine Yf and p_1; then

$$Zf \equiv s x_1^{(s-1)} p_1 + \overline{y}_2^{(s-1)} p_2 + \overline{y}_3^{(s-1)} p_3 + \overline{y}_4^{(s-1)} p_4$$

must occur. Now, combine Yf and Zf, and there results a transformation of the form

$$-s x_1^{2s-2} p_1 + \overline{\xi}_2^{(2s-2)} p_2 + (\,)\, p_3 + (\,)\, p_4 .$$

But the sth is the maximum order; hence

$$2s - 2 < s + 1 \quad \therefore \quad s < 3 .$$

Similarly if $\xi_1^{(s)} \equiv 0$ and $\xi_2^{(s)}$ be not 0, then also $s < 3$.

3. *To find the transformations of the second order.*

Let

$$\sum_1^4 {}_i\, \xi_i^{(2)} p_i$$

be such a transformation.

If now $\xi_2^{(2)}$ and $\xi_4^{(2)}$ are not both identically zero, they can only be functions of x_3, x_4; and $\xi_1^{(2)}$, $\xi_2^{(2)}$ must contain only x_1, x_2.

(a) If $\xi_3^{(2)} \equiv \xi_4^{(2)} \equiv 0$, then $\xi^{(2)}$ and $\xi^{(2)}$ must evidently be both different from zero, and by the bracket operation it easily follows that the transformations of the second order are of the forms

$$Xf \equiv x_1 x_2 p_1 + x_2^2 p_2 , \quad Yf \equiv x_1^2 p_1 + x_1 x_2 p_2 .$$

(b) If $\xi_3^{(2)}$ and $\xi_4^{(2)}$ are not both identically zero, since the variables x_1, x_2 and x_3, x_4 are equally privileged, it may be assumed that also $\xi_1^{(2)}$ and $\xi_2^{(2)}$ are not zero. Therefore there are two transformations of the form

$$X_1 f \equiv x_1^2 p_1 + x_1 x_2 p_2 + \xi_3^{(2)}(x_3, x_4)\, p_3 + \xi_4^{(2)}(x_3, x_4)\, p_4 ,$$

and

$$Y_1 f = x_1 x_2 p_1 + x_2^2 p_2 + y_3^{(2)}(x_3, x_4)\, p_3 + y_4^{(2)}(x_3, x_4)\, p_4 .$$

By combining $X_1 f$ with $x_1 p_2$, and $Y_1 f$ with $x_2 p_1$, it is seen that also in this case Xf and Yf must occur alone. In like manner

$$x_3 x_4 p_3 + x_4^2 p_4 , \quad x_3^2 p_3 + x_4 x_3 p_4$$

may occur alone; and these forms are all the possible transformations of the second order that can occur in this case.

II.—CASE IN WHICH NO TRANSFORMATIONS OF THE SECOND ORDER OCCUR.

This case divides itself again according to whether the transformations

$$U = \sum_1^4 {}_i x_i p_i ,$$

and

$$T' = x_1 p_1 + x_2 p_2 - x_3 p_3 - x_4 p_4$$

occur alone, or additively, i. e. in the form

$$\alpha . U + \beta . T . \qquad (\alpha, \beta = \text{const.})$$

1. *U and T occurring alone.*

The initial terms of the transformations are

$$p_1, \; p_2, \; p_3, \; p_4,$$

$$x_1 p_2, \; x_1 p_1 - x_2 p_2, \; x_2 p_1 ; \quad x_3 p_4, \; x_3 p_3 - x_4 p_4, \; x_4 p_3 \qquad (S_i)$$

$$U, \; T.$$

Here we find at once the relations

$$(S_1, S_2) = -2S_1 , \; (S_1, S_3) = S_2 , \; (S_2, S_3) = -2S_3 , \; (S_i, U) = 0 ,$$

$$(S_4, S_5) = -2S_4 , \; (S_4, S_6) = S_5 , \; (S_5, S_6) = -2S_6 , \; (S_i, T) = 0 ;$$

$$(U, T) = 0 .$$

We wish to normalize now with the transformation U, in order to find the setting of our group.

We know that

$$(p_1, U) = p_2 + \sum_1^6 {}_i a_i S_i + \beta U + \gamma T , \qquad (a_i, \beta, \gamma = \text{const.})$$

We may introduce a new p_1 by the substitution

$$p_1 = p_1 + \sum_1^6 {}_i A_i S_i + BU + GT , \quad (A_i, B, G = \text{const.})$$

Then we find

$$(p_1, U) = p_1 + \sum_1^6{}_i a_i S_i + \beta U + \gamma T - \sum_1^6{}_i A_i S_i - BU - GT.$$

Since the A_i, B, and G are arbitrary constants, we can choose

$$A_i = a_i, \quad B = \beta, \quad G = \gamma,$$

and, therefore,

$$(p_1, U) = \bar{p}_1;$$

or, as nothing depends upon the symbol we use, we can write

$$(p_1, U) = p_1.$$

Thus p_1 and U are normally connected.

Proceeding analogously we can choose p_2, p_3, p_4 so that

$$(p_2, U) = p_2, \quad (p_3, U) = p_3, \quad (p_4, U) = p_4.$$

Let us now find the relations between the p_k and the S_i and T. We have

$$(p_1, S_1) = p_2 + \sum_1^6{}_i a_i S_i + \beta U + \gamma T, \qquad (a_i, \beta, \gamma = \text{const.})$$

Form Jacobi's identity by means of p_1, S_1, and U; thus,

$$((p_1, S_1), U) + ((S_1, U), p_1) + ((U_1, p_1), S_1) = 0.$$

We find

$$(p_2, U) - (p_1, S_1) = 0$$

or

$$(p_1, S_1) = p_2.$$

This is a normal relation; and by proceeding analogously with all the transformations of the zero order, and those of the first order, we find that all the resulting relations are normal.

It only remains for us to find how the transformations of the zero order are connected among themselves.

We know that

$$(p_1, p_2) = \sum_1^4{}_i a_i p_i + \sum_1^6{}_k \beta_k S_k + \gamma U + \delta T,$$

where a_i, β_k, γ, δ are certain constants.

Form Jacobi's identity with p_1, p_2, and U; thus,

$$((p_1, p_2), U) + ((p_2, U), p_1) + ((U, p_1), p_2) = 0,$$

or

$$\sum_1^4{}_i a_i p_i - 2(p_1, p_2) = 0,$$

or

$$(p_1, p_2) \equiv 0 .$$

In the same way we find that the rest of the (p_i, p_k)'s are normal relations.

This gives us the group

$$\boxed{\begin{array}{c} p_1, p_2, p_3, p_4 ;\ x_1 p_2, x_1 p_1 - x_2 p_2, x_2 p_1 ;\ x_3 p_4, x_3 p_3 - x_4 p_4, x_4 p_3 ; \\ x_1 p_1 + x_2 p_2 - x_3 p_3 - x_4 p_4,\ x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4 . \end{array}}$$

But this group is *imprimitive*, since it contains two invariant subgroups, p_1, p_2 and p_3, p_4; that is, the two families of $\infty^2 M_2$: $\begin{cases} x_1 = c_1 \\ x_2 = c_2 \end{cases}$ and $\begin{cases} x_3 = c_3 \\ x_4 = c_4 \end{cases}$ (c_i const.) are invariant.

2. *U and T not occurring alone, but in the combined form $W = \alpha U + \beta T$.*

In this case, again, all the transformations of the first order are connected by normal relations. Let us normalize those of zero order by means of

$$S \equiv x_1 p_1 - x_2 p_2 + x_3 p_3 - x_4 p_4 ,$$

which is evidently a transformation of our group. As in (1), we can easily choose the $p_1, \ldots, p_4$ so that the following relations hold:

$$(p_1, S) \equiv p_1 , \quad (p_2, S) \equiv - p_2 \quad (p_3, S) \equiv p_3 , \quad (p_4, S) \equiv - p_4 .$$

We find now, without difficulty, by means of Jacobi's identity, that all the relations between the transformations of the zero and of the first orders are normal.

We find further, forming Jacobi's identity with

$$p_i, p_k, S ; \text{ and } p_i, p_k, S,$$

the following relations :

$$(p_1, p_2) \equiv a \,.\, W + b \,.\, S$$
$$(p_1, p_3) \equiv (p_2, p_4) \equiv 0$$
$$(p_1, p_4) \equiv e \,.\, W + d \,.\, S$$
$$(p_2, p_4) \equiv e \,.\, W + f \,.\, S$$
$$(p_3, p_4) \equiv m \,.\, W + n \,.\, S$$

where $a, \ldots, n$ are certain constants.

Now form Jacobi's identity with p_1, p_2, W; thus,

$$((p_1, p_2), W) + ((p_2, W), p_1) + ((W, p_1), p_2) \equiv 0 ,$$

and hence

$$(a + \beta)\, a \equiv (a + \beta)\, b \equiv 0\,.$$

Proceeding similarly with p_1, p_4, W, we find

$$a\,.\,c = a\,.\,d = 0\,;$$

and similarly with the other p_i, p_k, W, we find

$$a\,.\,e = a\,.\,f = 0\,,$$

$$a\,(a - \beta) = b\,, \quad e\,(a + \beta) = f\,, \quad c\,(a + \beta) = -\,d\,,$$

$$m\,(a + \beta) = -\,n\,, \quad c\,(a - \beta) = -\,d\,, \quad e\,(a - \beta) = -\,f\,.$$

Hence,

$$f = e\,.\,\beta = d = c\,.\,\beta = 0\,.$$

(A) *Suppose now* $a \gtrless 0$, then $c = d = 0$. If also $a \gtrless -\beta$ we have $a = b = 0$; and we see at a glance that p_1, p_2 must form an invariant subgroup; and the group interests us no longer, as it must be imprimitive.

If $a = -\beta$, we have $m = n = 0$; and then p_3, p_4 form an invariant subgroup. Thus we have no possible primitive group when $a \gtrless 0$.

(B) *Suppose* $a = 0$. Then if $\beta \gtrless 0$, all our other constants are zero, and we find a group

$$p_1, p_2, p_3, p_4\,;\; x_1 p_2, x_1 p_1 - x_2 p_2, x_2 p_1\,;\; x_3 p_4, x_3 p_3 - x_4 p_4, x_4 p_3\,;$$
$$T \equiv x_1 p_1 + x_2 p_2 - x_3 p_3 - x_4 p_4\,.$$

But this group is evidently imprimitive.

If $\beta = 0$, then no transformation W occurs at all; and we get the above group again without the transformation T. This group is also evidently imprimitive.

III.—Cases in which transformations of the second order can occur.

1. $\xi_3^{(2)}$ *and* $\xi_4^{(2)}$ *identically zero.*

Then the transformations of the second order have the forms,

$$Xf \equiv x_1 x_2 p_1 + x_2^2 p_2\,, \quad Xf \equiv x_1^2 p_1 + x_1 x_2 p_2\,.$$

We have to make another subdivision, now, according to whether

$$U \equiv \sum_1^4 x_i p_i\,, \text{ and } T \equiv x_1 p_1 + x_3 p_3 - x_4 p_4 + x_2 p_2$$

occur free or not.

(a) If U and T occur free, we have, besides these transformations and Xf and Yf,

$$p_1, p_2, p_3, p_4 ; \; x_1 p_2, x_1 p_1 - x_2 p_2, x_2 p_1 ; \; x_3 p_4, x_3 p_3 - x_4 p_4, x_4 p_3 . \qquad (S_i)$$

We shall now find how these transformations are connected.

We have

$$(S_1, U) \equiv \alpha Xf + \beta Yf , \qquad (\alpha, \beta \text{ constants.})$$

Now introduce a new S_1 by means of

$$S_1 \equiv S_1 - \alpha Xf - \beta Yf ,$$

and we find

$$(S_1, U) \equiv 0 ;$$

or, as we may write it,

$$(S_1, U) \equiv 0 .$$

In like manner we may choose the other S_i so that

$$(S_i, U) \equiv 0 . \qquad (i = 2, \ldots, 6)$$

By means of Jacobi's identity we can now show that all the transformations of the first order are connected by normal relations. Thus, for example, we know that

$$(S_1, S_2) \equiv aXf + bY - 2S ,$$

and

$$((S_1, S_2) U) + ((S_2, U) S_1) + ((U, S_1) S_2) = 0 ;$$

that is,

$$a = b = 0 .$$

We can easily choose the p_i's so that

$$(p_i, U) \equiv p_i ; \; i = 1, \ldots, 4 ;$$

and we find from Jacobi's identity that the p_i's are connected by normal relations with the other transformations of the first and second orders as well as with each other. This gives us the (imprimitive) group,

$$p_1, p_2, p_3, p_4 ; \; x_1 p_2, x_1 p_1 - x_2 p_2, x_2 p_1 ; \; x_3 p_4, x_3 p_3 - x_4 p_4, x_4 p_3 ;$$
$$x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4, x_1 p_1 + x_2 p_2 - x_3 p_3 - x_4 p_4 ; \; x_1 x_2 p_1 + x_2^2 p_2,$$
$$x_1^2 p_1 + x_1 x_2 p_2 .$$

(b). If the transformations U and T do not occur free they only occur in the form,

$$aU + \beta T. \qquad (a, \beta \text{ constants})$$

Our transformations are those which we have above designated as p_i, S_k, Xf, Yf, and $aU + \beta T$.

Let us combine p_2 and Xf; thus,

$$(p_2, Xf) = x_1 p_1 + 2x_2 p_2 .$$

Also,

$$(p_1, Yf) = 2x_1 p_1 + x_2 p_2 .$$

By addition, we see that a transformation of the former $x_1 p_1 + x_2 p_2$ must occur. That is, we must have

$$x_1 p_1 + x_2 p_2 = \rho . S_2 + \sigma (aU + \beta T) ; \qquad (\rho, \sigma \text{ constants})$$

and hence

$$\rho = 0 , \quad a = \beta , \quad \sigma = 1 .$$

The transformation $aU + \beta T$ has therefore the form,

$$x_1 p_1 + x_2 p_2 .$$

Let us normalize our relations, now, with the transformation

$$S = x_1 p_1 + x_2 p_2 + x_3 p_3 - x_4 p_4 .$$

We notice that

$$(Xf, S) = - Xf , \quad (Yf, S) = - Yf ;$$

and we can easily choose the S_k such that

$$(S_1, S) = 0 , \quad (S_2, S) = 0 , \quad (S_3, S) = 0 ,$$
$$(S_4, S) = - 2S_4 , \quad (S_5, S) = 0 , \quad (S_6, S) = 2S_6 .$$

It is easy to see that all the (S_i, S_k) are normal; and we can choose the p_i, in the usual manner, so that,

$$(p_1, S) = p_1 , \quad (p_2, S) = p_2 , \quad (p_3, S) = p_3 ,$$
$$(p_4, S) = - p_4 + aXf + bYf . \qquad (a, b \text{ constants})$$

We have

$$(p_4, S_2) = \sum_1^6 a_i S_i + aT + \beta_2 Xf + \gamma_2 Yf ;$$

and by Jacobi's identity,

$$- 2a_4 S_4 + 2a_6 S_6 - \beta_2 Xf - \gamma_2 Yf + \sum_1^6 a_i S_i + aT$$
$$+ \beta_2 Xf + \gamma_2 Yf + aXf + bYf = 0 ,$$

or

$$a_i = a = b = a = 0 .$$

Hence

$$(p_4, S) = - p_4 .$$

We find, by proceeding analogously, that all

$$(p_i, S_k) \qquad (i, k = 1, 2, 3)$$

are normal; as are also (p_i, Xf) and (p_i, Yf), $i = 1, \ldots, 4$.

Further,

$$(p_1, S_5) = (p_1, S_6) = (p_2, S_5) = (p_2, S_6) = (p_3, S_6) = (p_4, S_4) = 0 ,$$

and

$$(p_3, S_5) = p_3 , \quad (p_4, S_6) = p_3 ;$$

Also,

$$(p_1, p_2) = (p_1, p_3) = (p_2, p_3) = 0 .$$

By repeated applications of Jacobi's identity, we find moreover,

$$\begin{cases} (p_4, S_2) = \beta_2 Xf + \gamma_2 Yf , \\ (p_4, S_1) = \beta_1 Xf + \gamma_1 Yf , \\ (p_4, S_3) = \beta_3 Xf + \gamma_3 Yf ; \end{cases}$$

($\beta_1, \ldots, \gamma_3$ constants)

and

$$\begin{cases} (p_1, S_4) = a_1 Xf + b_1 Yf , \\ (p_2, S_4) = a_2 Xf + b_2 Yf , \\ (p_3, S_4) = p_4 + a_3 Xf + b_3 Yf , \\ (p_4, S_5) = - p_4 + a_4 Xf + b_4 Yf . \end{cases}$$

($a_1, \ldots, b_4$ constants)

Analogously, we find,

$$\begin{cases} (p_1, p_4) = A_1 S_1 + A_2 S_2 + A_3 S_3 + A_5 S_5 + AS , \\ (p_2, p_4) = B_1 S_1 + B_2 S_2 + B_3 S_3 + B_5 S_5 + BS , \\ (p_3, p_4) = C_1 S_1 + C_2 S_2 + C_3 S_3 + C_5 S_5 + CS . \end{cases}$$

If, now, we form all Jacobian identities possible with the p_i, the S_k, S, Xf, and Yf, we will find, after a computation which we omit here on account of

its length,

$$\left\{\begin{array}{l} (p_1, p_4) \equiv -2B_2S_2, \\ (p_2, p_4) \equiv B_2S_2 - 3B_2(S_2, S_5), \\ (p_4, S_1) \equiv 2B_2Xf, \\ (p_4, S_2) \equiv -2B_2Yf, \\ (p_4, S_3) \equiv (p_1, S_4) \equiv (p_2, S_4) \equiv 0, \\ (p_3, S_4) \equiv p_4 + 2B_2Yf, \\ (p_4, S_5) \equiv -p_4 - 2B_2Yf. \end{array}\right.$$

Let us now introduce a new p_4 by means of the substitution

$$\bar{p}_4 \equiv p_4 + 2B_2Yf;$$

and, as this does not affect those transformations which have already been normalized, we see that this is equivalent to making $B_2 = 0$.

All our relations are now normal, and we find thus the group

$$\begin{array}{c} p_1, p_2, p_3, p_4;\ x_1p_2, x_1p_1 - x_2p_2, x_2p_1;\ x_3p_4, x_3p_3 - x_4p_4, x_4p_3; \\ x_1p_1 + x_2p_2, x_1x_2p_1 + x_2^2p_2, x_1^2p_1 + x_1x_2p_2. \end{array}$$

It is easy to see, however, that this group is imprimitive.

2. *Not all the $\xi_3^{(2)}$ and $\xi_4^{(2)}$ identically zero.*

Since the variables x_1, x_2 are equally privileged with x_3, x_4, we can also assume that not all $\xi_1^{(2)}$ and $\xi_2^{(2)}$ are identically zero. Thus we easily see that among the transformations of the second order must occur

$$X_1f \equiv x_1x_2p_1 + x_2^2p_2 + \xi_3^{(2)}(x_3, x_4)\,p_3 + \xi_4^{(2)}(x_3x_4)\,p_4,$$

and

$$X_2f \equiv x_1^2p_1 + x_1x_2p_2 + \eta_3^{(2)}(x_3x_4)\,p_3 + \eta_4^{(2)}(x_3, x_4)\,p_4.$$

Now combine X_1f and x_1p_2; hence

$$X_1f \equiv x_1^2p_1 + x_1x_2p_2$$

must occur alone; so also must evidently

$$X_2f \equiv x_1x_2p_1 + x_2^2p_2.$$

Then, since the variables are equally privileged, we see that

$$X_4f \equiv x_3x_4p_3 + x_4^2p_4,$$

and

$$X_3 f \equiv x_3^2 p_3 + x_3 x_4 p_4$$

must also occur alone ; and these are all possible transformations of the second order.

If now, with the p_i, the S_k, and the $X_j f$,

$$U \equiv \sum_1^4 x_i p \text{ and } x_1 p_1 + x_2 p_2 - x_3 p_3 - x_4 p_4 \equiv T$$

occur alone, we easily find the (imprimitive) group

$$\boxed{\begin{array}{c} p_1, p_2, p_3, p_4 ; x_1 p_2, x_1 p_1 - x_2 p_2, x_2 p_1 ; x_3 p_4, x_3 p_3 - x_4 p_4, x_4 p_3 ; \\ x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4 , \ x_1 p_1 + x_2 p_2 - x_3 p_3 - x_4 p_4 ; \\ x_1^2 p_1 + x_1 x_2 p_2, x_1 x_2 p_1 + x_2^2 p_2 ; x_3^2 p_3 + x_3 x_4 p_4, x_3 x_4 p_3 + x_4^2 p_4 . \end{array}}$$

If, however, U and T only occur in the combination

$$\alpha U + \beta T, \qquad (\alpha, \beta \text{ constants})$$

our transformations will not form a group at all. For

$$\tfrac{1}{3}\{(p_2, X_2 f) + (p_1, X_1 f)\} \equiv x_1 p_1 + x_2 p_2 ,$$

or

$$x_1 p_1 + x_2 p_2 \equiv \rho (x_1 p_1 - x_2 p_2) + \sigma (x_3 p_3 - x_4 p_4) + \nu (\alpha U + \beta T). \qquad (\rho, \sigma, \nu \text{ constants})$$

Thus we see

$$\rho \equiv \sigma \equiv 0 ,$$
$$\nu (\alpha - \beta) \equiv 0 .$$

In order to have a group, therefore, we must have $\alpha = \beta$, and

$$\alpha U + \beta T \equiv x_1 p_1 + x_2 p_2 .$$

But

$$\tfrac{1}{3}\{(p_1, X_1 f) + (p_3, X_3 f)\} \equiv x_3 p_3 + x_4 p_4$$

must belong to our group, if it is a group ; and we see at once that this transformation cannot have the necessary form

$$\sum_1^6 a_i S_i + a (x_1 p_1 + x_2 p_2) . \qquad (a_i, a \text{ constants})$$

Therefore, in this case, our transformations do not form any group.

Hence we have shown that there are no primitive groups of infinitesimal transformations in space of four dimensions which leave two different manifoldnesses of two dimensions invariant.

ON THE ROOTS OF THE BESSEL AND CERTAIN RELATED FUNCTIONS.

By Prof. James McMahon, Ithaca, N. Y.

I.—Roots of $J_n(x)$.

The formulas of Sir G. G. Stokes for the roots of $J_0(x)$, $J_1(x)$ in the form of descending series are well known.* It is first proposed to obtain a similar expression for the roots of $J_n(x)$, n being unrestricted.

1. *Use of descending series for $J_n(x)$, with an auxiliary angle θ.*

Take the descending series†

$$\sqrt{\tfrac{1}{2}\pi x}\,J_n(x) = \cos(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi)\,\varphi_n(x) + \sin(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi)\,\psi_n(x), \qquad (1)$$

in which

$$\varphi_n(x) = 1 - \frac{(1^2 - 4n^2)(3^2 - 4n^2)}{2!\,(8x)^2} + \frac{(1^2 - 4n^2)(3^2 - 4n^2)(5^2 - 4n^2)(7^2 - 4n^2)}{4!\,(8x)^4} - \ldots,$$

$$\psi_n(x) = \frac{1^2 - 4n^2}{8x} - \frac{(1^2 - 4n^2)(3^2 - 4n^2)(5^2 - 4n^2)}{3!\,(8x)^3} + \ldots;$$

and let

$$\varphi_n(x) = R\cos\theta, \quad \psi_n(x) = R\sin\theta;$$

then

$$\sqrt{\tfrac{1}{2}\pi x}\,J_n(x) = R\cos(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta), \qquad (2)$$

and the equation $J_n(x) = 0$ becomes

$$\cos(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta) = 0;$$

whence

$$x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta = \tfrac{1}{2}(2s - 1)\pi,$$

s being any integer.

$$\therefore \quad x = \beta + \theta, \qquad (3)$$

where

$$\beta = \tfrac{1}{4}\pi(2n + 4s - 1). \qquad (4)$$

2. *Series for $\tan\theta$ in powers of $\dfrac{1}{8x}$.*

Let $\frac{1}{8}x^{-1} = y$, $4n^2 = m$; then, from (2) and (1),

$$\tan\theta = -(m-1)\,y\,\frac{1 - by^2 + dy^4 - \ldots}{1 - ay^2 + cy^4 - \ldots}$$

$$= -(m-1)\,y\{1 + (a-b)\,y^2 + [a(a-b) + d - c]\,y^4 + \ldots\},$$

* Camb. Phil. Trans., Vol. IX, or Math. and Phys. Papers, Vol. II, pp. 353, 355.

† Annals of Mathematics, Vol. VIII, No. 2, p. 60.

in which

$$a = \tfrac{1}{2}(m-1)(m-9), \quad b = \tfrac{1}{6}(m-9)(m-25),$$

$$c = \tfrac{1}{24}(m-1)(m-9)(m-25)(m-49),$$

$$d = \tfrac{1}{120}(m-9)(m-25)(m-49)(m-81),$$

and

$$a - b = \tfrac{1}{3}(m^2 + 2m - 99),$$

$$a(a-b) + d - c = \tfrac{2}{15}(m-9)(m^3 + 15m^2 - 81m - 5695);$$

$$\therefore \tan\theta = -(m-1)y - \tfrac{1}{3}(m-1)(m^2+2m-99)y^3$$

$$- \tfrac{2}{15}(m-1)(m-9)(m^3+15m^2-81m-5695)y^5 - \ldots. \qquad (5)$$

3. *Series for* θ, $(= x - \beta)$, *in powers of* $\dfrac{1}{8x}$.

Since

$$\theta = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \ldots,$$

and $\tan\theta$ is of the form

$$Ay + By^3 + Cy^5 + \ldots,$$

$$\therefore \quad \theta = Ay + (B - \tfrac{1}{3}A^3)y^3 + (C - A^2B + \tfrac{1}{5}A^5)y^5 + \ldots,$$

which, on replacing A, B, C by the coefficients in (6) and reducing, becomes

$$\theta = -(m-1)y - \tfrac{4}{3}(m-1)(m-25)y^3$$

$$- \tfrac{32}{5}(m-1)(m^2-114m+1073)y^5 - \ldots; \qquad (6)$$

$$\therefore x = \beta - \frac{m-1}{8x} - \frac{4(m-1)(m-25)}{3(8x)^3}$$

$$- \frac{32(m-1)(m^2-114m+1073)}{5(8x)^5} - \ldots. \qquad (7)$$

4. *Series for* x *in powers of* $\dfrac{1}{\beta}$.

From an equation of the form

$$x = \beta + \frac{p}{x} + \frac{q}{x^3} + \frac{r}{x^5} + \ldots,$$

Lagrange's theorem gives

$$x = \beta + \frac{p}{\beta} + \frac{q-p^2}{\beta^3} + \frac{r-4pq+2p^3}{\beta^5} + \ldots.$$

Substituting for p, q, r, the coefficients $-\frac{1}{8}(m-1)$, $-\frac{4}{3\cdot 8^3}(m-1)(m-25), \ldots$, the reduced result may be written

$$x_n^{(s)} = {}_i\beta - \frac{m-1}{8\,{}_i\beta} - \frac{4(m-1)(7m-31)}{3\,(8\,{}_i\beta)^3} - \frac{32(m-1)(83m^2-982m+3779)}{15\,(8\,{}_i\beta)^5} - \ldots,^* \qquad (8)$$

in which $x_n^{(s)}$ represents the sth root (in order of magnitude) of the equation $J_n(x) = 0$, while ${}_i\beta$ stands for $\frac{1}{4}\pi(2n-1+4s)$, and m is $4n^2$.

II.—Roots of the first derivative of $J_n(x)$.†

5. *Descending series for $J_n(x)$.*

Divide equation (1) by $x^{\frac{1}{2}}$, differentiate, multiply by $2x^{\frac{1}{2}}$, and rearrange; then

$$\sqrt{2\pi x}\,J_n(x) = M\cos(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi) - N\sin(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi), \qquad (9)$$

in which

$$M = \frac{2A_1 - A_0}{x} - \frac{2A_3 - 5A_2}{x^3} + \frac{2A_5 - 9A_4}{x^5} - \ldots,$$

$$N = 2A_0 - \frac{2A_2 - 3A_1}{x^2} + \frac{2A_4 - 7A_3}{x^4} - \ldots,$$

$A_0, A_1, -A_2, -A_3, \ldots$ being the coefficients of successive powers of x^{-1} in equation (1). The simplified expressions are

$$\left.\begin{aligned} M &= -(2m+6)\,y + \tfrac{1}{3}y^3(1+25-341+315) \\ &\quad - \tfrac{1}{60}y^5(1+15-6342+182510-1267659+1091475) + \ldots, \\ N &= 2 - y^2(m^2+14m-15) \\ &\quad + \tfrac{1}{12}y^4(1+28-1946+16092-14175) - \ldots, \end{aligned}\right\} \qquad (10)$$

* As it may sometimes be desirable to continue the series one step further, it is perhaps worth while to record that the next term is

$$-\frac{64(m-1)(6949m^3-153855m^2+1585743m-6277237)}{105\,(8\,{}_i\beta)^7}.$$

When n is 0, or 1, equation (8) reduces to Stokes's formulas; and when n is $\frac{3}{2}$ it gives the series for the roots of $J_{\frac{3}{2}}(x)$ obtained independently by Lord Rayleigh (Theory of Sound, Vol. I, p. 279).

In Professor Stokes's second formula the numerator 0.245835 should be 0.245270, a mistake which evidently arose from overlooking a "star" in the table of logarithms.

† The equation $J_n'(x) = 0$ is of hardly less importance in applied mathematics than the equation $J_n(x) = 0$. See Rayleigh's Theory of Sound, Vol. II, p. 266, Lamb's Hydrodynamics, p. 197, J. J. Thomson's Recent Researches in Electricity and Magnetism, p. 346.

in which m stands for $4n^2$, and y for $\frac{1}{8}x^{-1}$, the coefficients of the descending powers of m being "detached" for convenience.

Next, let an auxiliary angle θ' be defined by the equations

$$M = s \sin \theta', \quad N = s \cos \theta'; \tag{11}$$

then

$$\sqrt{2\pi x}\, J_n(x) = -s \sin (x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta'), \tag{12}$$

and the equation $J_n(x) = 0$ becomes, as before,

$$x = {}_1\beta' + \theta', \tag{13}$$

where

$${}_1\beta' = \tfrac{1}{4}\pi(2n + 4s + 1).$$

6. *Series for tan θ'.*

By (11) and (10), the reduced expression is

$$\tan \theta' = -y(m+3) - \tfrac{1}{3}y^3(1 + 13 + 211 - 225)$$
$$- \tfrac{2}{15}y^5(1 + 25 + 658 + 11370 - 97779 + 85725) - \ldots. \tag{14}$$

7. *Series for θ', $(= x - {}_1\beta')$, in powers of y.*

The method of Art. 3 gives

$$x = {}_1\beta' - \frac{m+3}{8x} - \frac{4(1 + 46 - 63)}{3(8x)^3}$$
$$- \frac{32(1 + 185 - 2053 + 1899)}{5(8x)^5} \ldots. \tag{15}$$

8. *Series for x in powers of $\dfrac{1}{8\,{}_1\beta'}$.*

Applying Lagrange's theorem as in Art. 4, the final result is

$$x_n^{(s)} = {}_1\beta' - \frac{m+3}{8\,{}_1\beta'} - \frac{4(7m^2 + 82m - 9)}{3(8\,{}_1\beta')^3}$$
$$- \frac{32(83m^3 + 2075m^2 - 3039m + 3537)}{15(8\,{}_1\beta')^5} \ldots;^* \tag{16}$$

in which $x_n^{(s)}$ represents the sth root (in order of magnitude) of the equation $J_n(x) = 0$, m is $4n^2$, and ${}_1\beta'$ is $\frac{1}{4}\pi(2n + 4s + 1)$.

* It will be found that $n = 0$ in (16) gives the same result as $n = 1$ in (8), in accordance with the identity $J_0(x) = -J_1(x)$.

A further test is furnished by the fact that when $n = \frac{1}{2}$, the equation $J_n(x) = 0$ is equivalent to $\tan x = 2x$, as may be seen from (9), and that the independent solution of the latter equation agrees with (16). Similarly when $n = \frac{3}{2}$.

III.—Roots of the derivative of $x^{-\frac{1}{2}}J_n(x)$.*

9. It will be found that the M and N of Art. 5 are now to be replaced by

$$M' = \frac{A_1 - A_0}{x} - \frac{A_3 - 3A_2}{x^3} + \frac{A_5 - 5A_4}{x^5} - \ldots,$$

$$N' = A_0 - \frac{A_2 - 2A_1}{x^2} + \frac{A_4 - 4A_3}{x^4} - \ldots;$$

and that the series for $\tan\theta'$ becomes

$$\tan\theta'' = -y(m+7) - \tfrac{1}{3}y^3(1 + 25 + 427 - 453)$$
$$- \tfrac{2}{15}y^5(1 + 45 + 1378 + 23490 - 158819 + 133905) - \ldots;$$

whence, as in Arts. 3, 7, is derived the series for θ'', $(= x - \beta')$, giving

$$x = \beta' - \frac{m+7}{8x} - \frac{4(1 + 70 - 199)}{3(8x)^3} - \frac{32(1 + 245 - 3693 + 4471)}{5(8x)^5} - \ldots,$$

and finally, by Lagrange's theorem, as in Art. 4,

$$x_n^{(s)} = \beta' - \frac{m+7}{8\beta'} - \frac{4(7m^2 + 154m + 95)}{3(8\beta')^3}$$
$$- \frac{32(83m^3 + 3535m^2 + 3561m + 6133)}{15(8\beta')^5} - \ldots.\dagger \qquad (17)$$

IV.—Roots of the Second Bessel Function, and of its Derivative.

10. Take the descending series‡

$$\sqrt{\frac{2x}{\pi}}\,K_n(x) = \cos(x - \tfrac{3}{4}\pi - \tfrac{1}{2}n\pi)\,\varphi_n(x) + \sin(x - \tfrac{3}{4}\pi - \tfrac{1}{2}n\pi)\,\psi_n(x), \quad (18)$$

in which $\varphi_n(x)$, $\psi_n(x)$ are the same as in Art. 1 ;

$$\therefore \sqrt{\frac{2x}{\pi}}\,K_n(x) = R\cos(x - \tfrac{3}{4}\pi - \tfrac{1}{2}n\pi - \theta),$$
$$= -R\cos(x + \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta); \qquad (19)$$

* The roots of the equation $\frac{d}{dx}[x^{-\frac{1}{2}}J_n(x)] = 0$ determine the possible wave lengths of a gas vibrating within a spherical envelope (Theory of Sound, Vol. II, p. 231).

† As a test, $n = \frac{1}{2}$ in (17) gives the same result as $n = \frac{3}{2}$ in (8), both being the solution of $\tan x = x$; and $n = \frac{3}{2}$ in (17) gives correctly the roots of $(2 - x^2)\tan x = 2x$, referred to on pp. 232-3, Theory of Sound, Vol. II.

‡ Annals of Mathematics, Vol. VIII, No. 2, p. 61.

and the equation $K_n(x) = 0$ gives

$$\cos(x + \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta) = 0;$$

$$\therefore \quad x + \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta = \tfrac{1}{2}(2s-1)\pi;$$

$$\therefore \quad x = \beta' + \theta,$$

where $$\beta' = \tfrac{1}{4}\pi(2n + 4s - 3),$$

and θ is the same as in Sect. I. Hence the series in (8), which gives the roots of $J_n(x)$, gives also the roots of $K_n(x)$, when β', $(= \beta - \tfrac{1}{2}\pi)$, is put for β.

11. *Derivative of* $K_n(x)$.

In the same way as in Art. 5 equations (9), (12) were derived from (1), so from (18) it follows that

$$2\sqrt{\frac{2x}{\pi}}\,K'_n(x) = -S\sin(x - \tfrac{3}{4}\pi - \tfrac{1}{2}n\pi - \theta')$$
$$= S\sin(x + \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta'); \qquad (20)$$

$\therefore$ the series in (16) gives also the roots of $K'_n(x)$, if β' be diminished by $\tfrac{1}{2}\pi$.

V.—Roots of the complete Bessel function, and of its derivative.

12. In a large class of physical problems,* k is to be determined so that the complete function

$$AJ_n(kr) + BK_n(kr)$$

may vanish on the concentric circumferences $r = a$, $r = b$. In this case

$$AJ_n(ka) + BK_n(ka) = 0,$$

$$AJ_n(kb) + BK_n(kb) = 0;$$

$$\therefore \quad \frac{K_n(ka)}{J_n(ka)} = \frac{K_n(kb)}{J_n(kb)}.$$

Let $a = \rho b$, $\rho > 1$, and $kb = x$;

$$\therefore \quad \frac{K_n(x)}{J_n(x)} = \frac{K_n(\rho x)}{J_n(\rho x)}; \qquad (21)$$

but

$$\frac{K_n(x)}{J_n(x)} = \frac{\pi}{2}\tan(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta), \qquad (22)$$

by (2) and (19);

$$\therefore \quad \tan(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta) = \tan(\rho x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta_1),$$

* See Byerly's "Fourier Series, etc.," pp. 230, 231, 234.

$$\therefore \quad \rho x - \theta_1 = x - \theta + s\pi\,,$$

s being any integer. Substitute for θ the series in (6), and for θ_1 use the same series with x replaced by ρx;

$$\therefore \quad x = \frac{s\pi}{\rho - 1} + \frac{m-1}{8\rho x} + \frac{4\,(m-1)\,(m-25)\,(\rho^3-1)}{3\,(8\rho x)^3\,(\rho-1)}$$
$$+ \frac{32\,(m-1)\,(m^2-114m+1073)\,(\rho^5-1)}{5\,(8\rho x)^5\,(\rho-1)} + \ldots; \qquad (23)$$

whence, as in Art. 4,

$$x_n^{(s)} = \beta + \frac{p}{\beta} + \frac{q-p^2}{\beta^3} + \frac{r-4pq+2p^3}{\beta^5} + \ldots, \qquad (24)$$

wherein

$$\beta = \frac{s\pi}{\rho-1}\,,\ p = \frac{m-1}{8\rho}\,,$$

$$q = \frac{4\,(m-1)\,(m-25)\,(\rho^3-1)}{3\,(8\rho)^3(\rho-1)}\,,\ r = \frac{32\,(m-1)(m^2-114m+1073)(\rho^5-1)}{5\,(8\rho)^5\,(\rho-1)}\,.$$

13. In certain problems in fluid motion* the function

$$AJ_n(kr) + BK_n(kr)$$

is to be treated as in Art. 12; and the equation

$$\frac{K_n(x)}{J_n(x)} = \frac{K_n(\rho x)}{J_n(\rho x)}$$

is by (12) and (20) equivalent to

$$\tan\,(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta') = \tan\,(\rho x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta_1')\,.$$

Substitute for θ' the series in Art. 7, and for θ_1' the same with ρx used for x, and the result is of the same form as (24), with

$$p = \frac{m+3}{8\rho}\,,\quad q = \frac{4\,(m^2+46m-63)\,(\rho^3-1)}{3\,(8\rho)^3\,(\rho-1)}\,,$$

$$r = \frac{32\,(m^3+185m^2-2053m+1899)\,(\rho^5-1)}{5\,(8\rho)^5\,(\rho-1)}\,.$$

14. *Wave lengths of a gas bounded by concentric spherical surfaces.*

This problem is the same extension of Section III as the last is of Section II; hence the series for θ'' in Art. 9 is to take the place of that for θ' in

* As in computing the wave lengths of a gas or liquid bounded by two coaxal cylindrical surfaces.

Art. 7, and the result is again of the same form as (24), but

$$p = \frac{m+7}{8\rho}, \quad q = \frac{4\,(m^2+70m-199)\,(\rho^3-1)}{3\,(8\rho)^3\,(\rho-1)},$$

$$r = \frac{32\,(m^3+245m^2-3693m+4471)\,(\rho^5-1)}{5\,(8\rho)^5\,(\rho-1)}.$$

General Note.

Some of the foregoing series for $x_n^{(s)}$ may for certain values of n and the lower values of s not be sufficiently convergent to give a very close approximation to the corresponding root. In such cases better approximations to the earlier roots may be found by interpolation from appropriate tables.

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